Wall Crossing Bijections and Representations of Rational Cherednik Algebras

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Young diagrams

A *partition* λ is a weakly-decreasing list of positive integers

$$(\lambda_1, \lambda_2, \ldots, \lambda_k)$$

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$$

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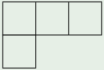
We represent a partition visually as a *Young diagram* whose row lengths correspond with part sizes.

Example

The Young diagram associated to partition $\lambda = (3, 1)$ is



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Symmetric bipartitions

A bipartition is a 2-tuple of partitions

$$\lambda = (\lambda^1, \lambda^2) = ((\lambda^1_1, \lambda^1_2, \ldots), (\lambda^2_1, \lambda^2_2, \ldots))$$

with size

$$|\lambda| = |\lambda^1| + |\lambda^2|.$$

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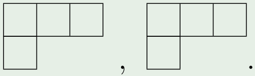
Definition

A bipartition is *symmetric* if $\lambda^1 = \lambda^2$.

Example

The bipartition $\lambda = ((3,1),(3,1))$ is a *symmetric bipartition*.



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A 2-charge $s = (s_1, s_2)$ is asymptotic for a given bipartition λ if $|s_1 - s_2| > |\lambda|$.

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The charge $s = (0, 9)$ is asymptotic for the bipartition $\lambda = ((3, 1), (3, 1))$.

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Definition

A *charged bipartition* is the data of a bipartition λ and a 2-charge s, written as $|\lambda, s\rangle$.

Finite-dimensionality

Remark

Charged bipartitions $|\lambda, s\rangle$ naturally label irreducible representations $L_{e,s}(\lambda)$ of $H_{|\lambda|,e,s}$. We call $|\lambda, s\rangle$ *finite-dimensional* if $L_{e,s}(\lambda)$ is finite-dimensional.

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Losev introduces *wall crossing bijections* to reduce the general problem of classifying finite-dimensional charged bipartitions to the case of asymptotic charge.

Theorem

Let $s_2 > s_1$ and suppose $s = (s_1, s_2)$ is asymptotic for bipartition λ. Then $\lambda^2 \neq \emptyset \implies |\lambda, s\rangle$ not finite-dimensional.

Example

The partition $|(\emptyset, (3)), (0, 4)\rangle$ is not finite-dimensional.

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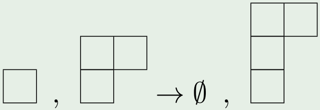
Example

Let $\lambda = ((1), (2, 1))$ and $s = (0, 2)$, then $\Phi^{\infty}_{(s_1,s_2)}(\lambda) = (\emptyset, (2, 1, 1))$.



$\Phi^{\infty}_{(s_1,s_2)}$ for asymptotic (s_1, s_2)

Remark

Consider a bipartition λ and a 2-charge (s_1, s_2). If $s_2 > |\lambda| + s_1$ is asymptotic, then $\Phi^{\infty}_{(s_1,s_2)}(\lambda) = \lambda$.

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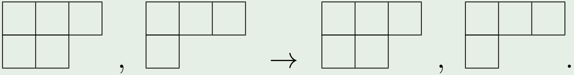
Example

Let $\lambda = ((3,1),(3,1))$ and $s = (0,10)$.



$\Theta_{e,s}$ as composition of consecutive $\Phi^{\infty}_{(s_1,ke+s_2)}$

Definition

Let $e > 1$ be an integer and $s = (s_1, s_2)$ a 2-charge. Define

$$\Theta_{e,s} = \prod_{k=0}^{\infty} \Phi^{\infty}_{(s_1,ke+s_2)} = \cdots \circ \Phi^{\infty}_{(s_1,Ne+s_2)} \circ \cdots \circ \Phi^{\infty}_{(s_1,e+s_2)} \circ \Phi^{\infty}_{(s_1,s_2)}$$

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Let λ be a bipartition, $e > 1$ an integer, and $s = (s_1, s_2)$ a 2-charge with $s_1 \leq s_2$.

Remark

Choose integer N so that charge $s' = (s_1, Ne + s_2)$ is asymptotic for λ. Then $|\Theta_{e,s}(\lambda), s'\rangle$ is finite-dimensional if and only if $|\lambda, s\rangle$ is finite-dimensional.

Main Question

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Let n and e be positive even integers, and let $s = (0, \frac{e}{2})$. Choose N for which $s' = (0, Ne + \frac{e}{2})$ is asymptotic.

Problem

Does there exist a nonempty symmetric bipartition λ such that the charged bipartition

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Does there exist a nonempty symmetric bipartition λ such that the charged bipartition

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Conjecture

No. In fact, the second component is always nonempty.

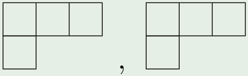
Conjecture

$|\Theta_{e,s}(\lambda), s'\rangle$ is not finite-dimensional for symmetric λ and $s = (0, \frac{e}{2})$.

Example

Let $\lambda = ((3,1),(3,1))$, $e = 2$, and $s = (0,1)$. Then

$$\Theta_{e,s}(\lambda) = \Phi^{\infty}_{(0,7)} \circ \Phi^{\infty}_{(0,5)} \circ \Phi^{\infty}_{(0,3)} \circ \Phi^{\infty}_{(0,1)}(\lambda).$$



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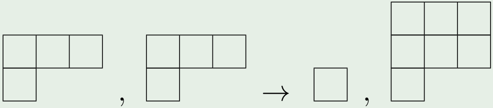

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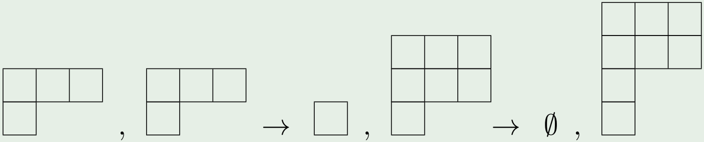

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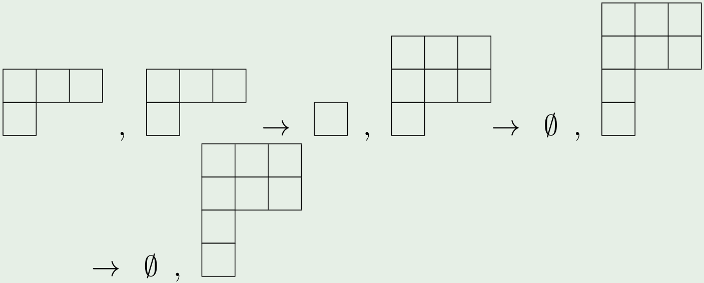

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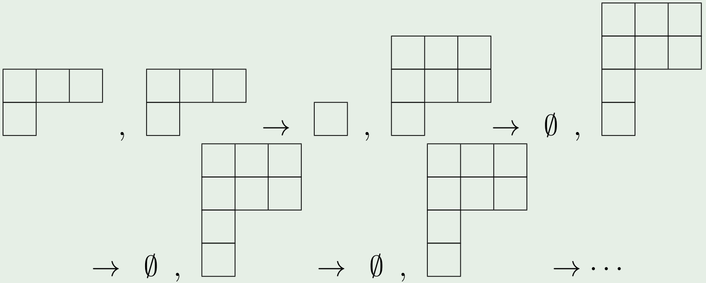

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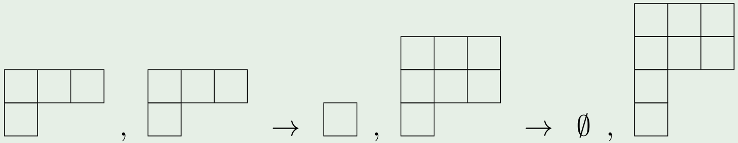
Thus $|\Theta_{2,(0,1)}((3,1),(3,1)),(0,1+2N)\rangle$ for $N \gg 0$ is not finite-dimensional.



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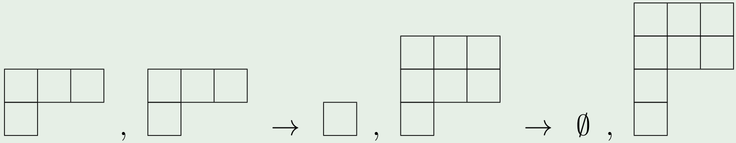

We have proved the conjecture in the case $e = 2$.

Theorem

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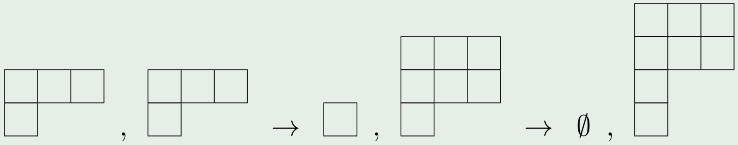

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For asymptotic s',

$$|\Theta_{2,(0,1)}((\lambda,\lambda)),s'\rangle = (\emptyset,(\lambda_1,\lambda_1,\lambda_2,\lambda_2,\ldots)).$$

$\Theta_{2,(0,1)}$ on $\lambda = ((4,3,1),(4,3,1))$

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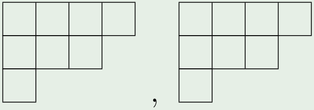

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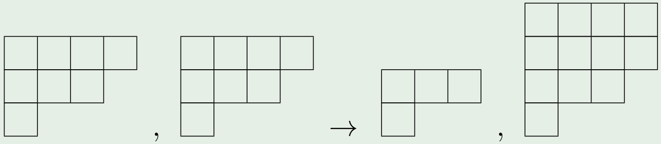

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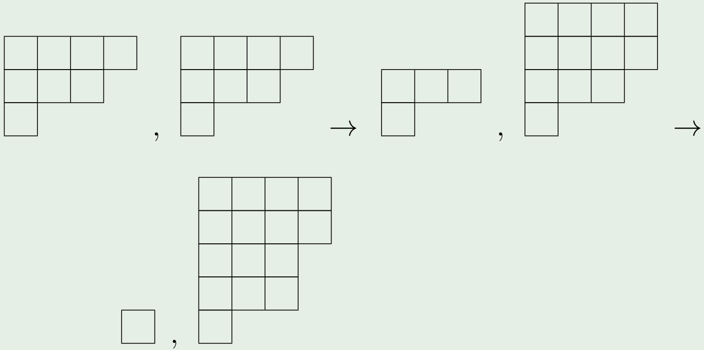

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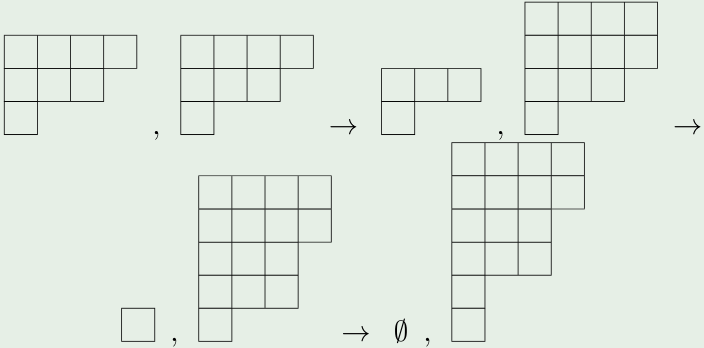

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We have verified the conjecture via computer for $|\lambda| < 50$. We are working toward a proof for general e.

Acknowledgements

I would like to thank

- My mentor Seth Shelley-Abrahamson
- The PRIMES-USA Program
- Head mentor Tanya Khovanova